1. The exam begins at 9:00 a.m. and ends at noon.
2. Calculators, books, notes, and communication with others are prohibited.
3. You must provide full justification for your answers.
4. For each problem, write your answer on a separate piece of paper and include your name and the number of the question at the top of the paper.
5. Omit one of the 15 questions by clearly marking through it with an X.

1. Find each limit, or explain why it does not exist:
   
   (a) \( \lim_{x \to 0} \frac{1 - \cos(x)}{x} \)
   
   (b) \( \lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} \)
   
   (c) \( \lim_{x \to \infty} \frac{x^4 + 7x^2 + 8}{e^x} \)

2. For the function \( f(x) = \frac{1}{x^2 + 1} \), sketch the graph and determine each of the following:
   
   (a) all local extrema (if any) and the type of each one;
   
   (b) the intervals on which the function is increasing, and the intervals on which it is decreasing;
   
   (c) the intervals on which the function is concave up, and the intervals on which it is concave down;
   
   (d) all inflection points (if any).

3. (a) State the official definition of the derivative. (It starts with “lim”)
   
   (b) Explain why the definition above corresponds to the slope of a tangent line.

4. Evaluate each integral:
   
   (a) \( \int \frac{\ln(x)}{x} \, dx \)
   
   (b) \( \int (x^3 + 5)^6 x^2 \, dx \)
   
   (c) \( \int x \cos(x) \, dx \)

5. Determine the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{3^n x^n}{n^2} \). Give reasons for your conclusions and identify the convergence test(s) used.

6. (a) State the Taylor series expansions about \( x = 0 \) for \( e^x \), \( \sin(x) \), and \( \cos(x) \).
   
   (b) Use your answers to (a) to show that the derivative of \( \sin(x) \) is \( \cos(x) \).

7. Consider the function \( f(x, y) = xy^2 - 6x^2 - 3y^2 \). Find all the critical points of \( f \) and classify each one (i.e., is it a local max, local min, or saddle point).

8. Compute \( \int_0^1 \int_{y/2}^1 e^{-x^2} \, dx \, dy \)

9. Consider the vector field \( \mathbf{F}(x, y) = (12x^2 + 3y^2 + 5x, 6xy - 3y^2 + 5x) \) (You may prefer to write this as \( \mathbf{F}(x, y) = (12x^2 + 3y^2 + 5y) \mathbf{i} + (6xy - 3y^2 + 5x) \mathbf{j} \)). Compute the line integral of \( \mathbf{F} \) over \( C \), where \( C \) is the unit circle, oriented counterclockwise, starting from the point \( (1, 0) \).
10. Prove or disprove: the set
\[
\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}
\]
is linearly independent.

11. You may assume that the matrices
\[
A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
are row equivalent.

(a) Find a basis for the null space of \( A \).
(b) Find a basis for the column space of \( A \).

12. Let \( A \) be a matrix with 7 rows and 9 columns, and let \( T_A \) denote the associated linear transformation, with respect to the canonical bases.

(a) Replace the question marks with something that makes sense: \( T_A : \mathbb{R}^7 \to \mathbb{R}^7 \).
(b) What are the possible values for the dimension of the range of \( T_A \). For each one, what is the corresponding dimension of the kernel?

13. Prove that, for any positive integer \( n \),
\[
1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

14. (a) Is it possible to have two sets \( A \) and \( B \) that satisfy all of the following conditions:
   - \( |A| = 3 \);
   - \( |B| = 4 \);
   - \( |A \cup B| = 5 \)?
   If so, give an example of such an \( A \) and \( B \). If not, explain why not.
   (b) Is it possible to find a function from \( \mathbb{N} \) to \( \mathbb{N} \times \mathbb{N} \) that is both one-to-one and onto? If so, do it. If not, explain why not.

15. Consider a relation \( R \) on \( \mathbb{Z} \) defined by: \( mRn \) if and only if \( 3m - n \) is even. Show that \( R \) is an equivalence relation. Describe the equivalence classes.