1. The exam begins at 9:00 a.m. and ends at noon.
2. Calculators, books, notes, and communication with others are prohibited.
3. You must provide full justification for your answers.
4. For each problem, write your answer on a separate piece of paper and include your name and the number of the question at the top of the paper.
5. Omit one of the 15 questions by clearly marking through it with an X.

1. Suppose it starts raining at noon and continues to rain (at varying levels of intensity) for the next six hours. Define a function \( R \) by

\[
R(t) = \text{the total rainfall (in inches) between noon and time } t.
\]

Assume \( t \) is measured in hours and that \( t = 0 \) corresponds to noon.

(a) Interpret the statement: \( R'(2) = 0 \).
(b) Is \( R'(t) \) always positive (for \( t \) between 0 and 6)? Explain.
(c) What would be the practical significance of an inflection point on the graph of \( R \)?

2. Compute the derivative of each function.

(a) \( f(x) = xe^{-2x} \)
(b) \( g(x) = \cos^2 \left( \frac{x}{x+1} \right) \)

3. A farmer wishes to build a rectangular pen, which will be subdivided into six smaller pens as shown. Given \( L \) feet of fence, what is the largest total area that can be enclosed?

4. State the definition of the definite integral, and describe some of its uses.

5. The functions pictured below are \( \sin(x) \) and \( \cos(x) \). Find the area of the shaded region.
6. State whether each series converges or diverges. State the test(s) you used to determine your answer.

(a) \[\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\]

(b) \[\sum_{n=1}^{\infty} \frac{n}{n^3 + 4}\]

(c) \[\sum_{n=1}^{\infty} \left(\frac{7}{8}\right)^n\]

7. For the function \(f(x, y) = x^2y^2 + y^3\), find:

(a) the gradient vector at the point \((1, 2)\);

(b) the directional derivative at the point \((1, 2)\) in the direction of the vector \(\langle 3, 4 \rangle\);

(c) the direction in which \(f\) is increasing most rapidly at the point \((1, 2)\).

8. Do one of the following two:

(a) Use a triple integral to show that the volume of a sphere of radius \(R\) is \(\frac{4}{3}\pi R^3\).

(b) Use a triple integral to show that the volume of a cylinder with radius \(R\) and height \(H\) is \(\pi R^2 H\).

9. Evaluate the line integral \(\int_{C} 3x^2 \sin(y) \, dx + x^3 \cos(y) \, dy\), where \(C\) is the curve defined by \(y = x^2\) from \(0 \leq x \leq \pi\).

[Alternate phrasing of the same question: Compute \(\int_{C} \vec{F} \cdot d\vec{r}\), where \(\vec{F}(x, y) = \langle 3x^2 \sin(y), x^3 \cos(y) \rangle\)]

10. \(A\) is a matrix with 8 rows and 10 columns.

(a) Fill in the blanks: \(A\) represents a linear transformation from Euclidean space of dimension to Euclidean Space of dimension .

(b) Make a table showing all possible dimensions of the null space of \(A\) and the corresponding dimensions of the column space of \(A\).

11. For the matrix \[
\begin{bmatrix}
1 & 2 & 1 & 2 \\
3 & 4 & 3 & 4 \\
0 & 1 & 0 & 1
\end{bmatrix}
\], find a basis for the null space and a basis for the column space.

12. \(A\) is an \(n \times n\) matrix. State as many conditions as you can which are equivalent to the statement “\(A\) is invertible.”

(b) Explain why an \(m \times n\) matrix \(B\) with \(m \neq n\) cannot be invertible.

13. Define a sequence \(\{a_n\}\) as follows:

- \(a_1 = 3\);
- \(a_2 = 4\);
- for \(n > 2\), \(a_n = a_{n-1} + a_{n-2}\)

Prove: for all integers \(n \geq 1\),

\[a_1 + a_2 + \cdots + a_n = a_{n+2} - a_2\]

14. Define a function \(f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}\) by \(f(x, y) = 4x - 2y\).

(a) Prove or disprove that \(f\) is one-to-one.

(b) Prove or disprove that \(f\) is onto.

15. Suppose that a certain college has eight math professors, five physics professors, and six CS professors.

(a) A committee is to be formed consisting of four of these 19 individuals. How many such committees are possible?

(b) A committee with a president, vice-president, and treasurer is to be formed consisting of four of these 19 individuals. How many such committees are possible?

(c) A committee is to be formed consisting of four of these 19 individuals, subject to the rule that at least one member from each subject must be represented. How many such committees are possible?