1. The exam begins at 9:00 a.m. and ends at noon.

2. Calculators, books, notes, and communication with others are prohibited.

3. You must provide full justification for your answers.

4. For each problem, write your answer on a separate piece of paper and include your name and the number of the question at the top of the paper.

5. Omit one of the 15 questions by clearly marking through it with an X.

1. For the function \( f(x) = x^3 - 3x^2 - 9x + 2 \),
   (a) find where the function is increasing and where it is decreasing;
   (b) find and classify all local extrema;
   (c) find where the function is concave up and where it is concave down;
   (d) find all inflection points;
   (e) sketch a detailed graph.

2. Find each limit, or show that it does not exist.
   (a) \( \lim_{x \to 0} \frac{\sin(2x)}{xe^x} \)
   (b) \( \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{\sin(x)} \right) \)

3. Evaluate each integral.
   (a) \( \int x \cos(x) \, dx \)
   (b) \( \int \frac{x^2}{(x + 1)^3} \, dx \)

4. (a) The region bounded by \( y = e^x \), \( y = 0 \), \( x = -1 \), and \( x = 1 \) is rotated around the \( x \)-axis. Sketch the resulting solid and find its volume.
   (b) The region bounded by \( y = x^2 \), \( y = 1 \), and the \( y \)-axis is rotated around the \( y \)-axis. Sketch the resulting solid and find its volume.

5. Find the interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(2x)^n}{n} \). Explain your reasoning with appropriate convergence tests.

6. Find the Taylor series for \( f(x) = e^x + e^{-x} \) about \( x = 0 \) (i.e. find the Maclaurin series).

7. Suppose \( f \) is a function of two variables, and suppose you know the following:
   (a) \( f_x(0,0) < 0 \)
   (b) \( f_y(0,0) > 0 \)
   (c) \( f_{yy}(0,0) < 0 \)

   Sketch a contour plot for \( f \) near \((0,0)\). Explain how the facts above manifest themselves in the plot.
8. Evaluate \( \int_0^1 \int_0^1 \sqrt{2 + x^3} \, dx \, dy \).

9. Consider the vector field \( \vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k} \), which you may prefer to write as \( \vec{F}(x, y) = (z, y, z) \).
   If \( S \) denotes the sphere of radius 2 centered at the origin, then the flux of \( \vec{F} \) out of \( S \) is given by the surface integral \( \int_S \vec{F} \cdot d\vec{A} \). Compute this integral.

10. Find a basis for the subset of \( \mathbb{R}^4 \) described by
    \[ \begin{cases} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{cases} : a, b, c \in \mathbb{R}, \]
    and state the dimension of this subspace.

11. (a) Row reduce the matrix
    \[
    \begin{bmatrix}
    1 & -2 & -5 \\
    -2 & 5 & 4 \\
    0 & 1 & -6
    \end{bmatrix}
    \]
    to reduced row echelon form.

   (b) Find the rank of \( A \) and the dimension of the null space of \( A \).

   (c) Find a basis for the row space of \( A \) and a basis for the column space of \( A \).

12. For the matrix
    \[
    A = \begin{bmatrix}
    2 & 3 \\
    4 & 1
    \end{bmatrix}
    \]
    find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = PDP^{-1} \).

13. Carefully prove that, for any integer \( n \geq 1 \), the quantity \( n^3 - n \) is divisible by 3.

14. Suppose we have the following sets:
    \[
    A = \{1, 2, 3, 4\} \\
    B = \{1, 2\} \\
    C = \{1, 2, 3, \ldots, 8\} \\
    D = \{1, 2, 3, \ldots, 16\}
    \]
    (a) Prove or disprove: there is a function from \( A \times B \) to \( C \) that is one-to-one and onto (\( \times \) denotes the Cartesian product).

   (b) Prove or disprove: there is a function from \( A \times B \times B \) to \( D \) that is one-to-one and onto.

15. For the problems below, define a word to be any ordered string of letters from the familiar 26-character alphabet.

    (a) How many 8-letter words have exactly four As?

    (b) How many 8-letter words have at least one A?

    (c) How many 8-letter words have at least five As?